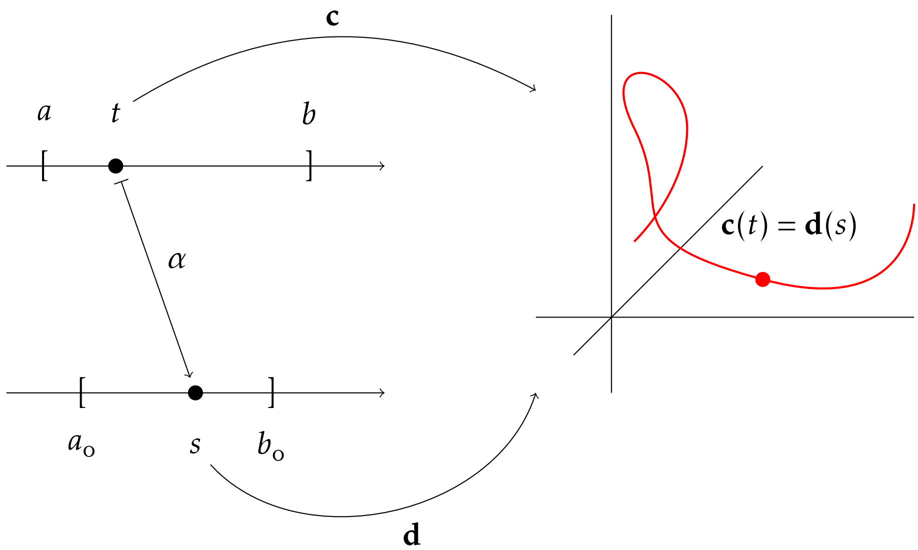
Calculus III, Problem Set 8, Q1a

October 2019

We are given a C^1 path $\mathbf{c}\colon [a,b] \to \mathbb{R}^3$ and a C^1, strictly increasing, bijective function $\alpha\colon [a,b] \to [a_0, b_0]$. From this we define a new C^1 path $\mathbf{d}\colon [a_0, b_0] \to \mathbb{R}^3$ by the formula $\mathbf{d}(s) :\equiv \mathbf{c}(t)$ for the unique t corresponding to s under α. Pictorially,



In order to make the definition of $\mathbf{d}$ we crucially make use of the fact that α is a bijection. With this in mind, we are then asked to show that the images of $\mathbf{c}$ and $\mathbf{d}$ are the same. In order to prove this, we'll have to remind ourselves what images *are*.

Given a function $f\colon D \subseteq \mathbb{R}^m \to \mathbb{R}^n$ the image of f, hereafter written $\text{img}(f)$, is the set of all points in the codomain $\mathbb{R}^n$ which the function "hits". Let's explore some examples.

Function type	Function Definition	Image
$\mathbb{R} \to \mathbb{R}$	$f(x) :\equiv x^2$	$[0, \infty)$
$\mathbb{N} \to \mathbb{N}$	$g(x) :\equiv 2x$	even numbers
$[0, \pi] \to \mathbb{R}^2$	$c(t) :\equiv (\cos t, \sin t)$	unit circle
$[0, 10\pi] \to \mathbb{R}^2$	$d(t) :\equiv (\cos t, \sin t)$	unit circle
$[-\pi, \pi] \to \mathbb{R}^2$	$e(t) :\equiv (\sin t, \cos t)$	unit circle

Table 1: Examples of functions and their images

Notice that images and graphs live in *different* places, for example, $\text{img}(f) \subseteq \mathbb{R}$ but the graph of f is a subset of $\mathbb{R}^2$. However, they are not unrelated – the image of a function is the result of taking its graph and throwing away the information relating to the input, that is, we record only the possible output values. While the images of the last three functions are all the same, their graphs are all different helicoids in $\mathbb{R}^3$, but the projections onto the output – those values visited – are all the same.

Now let's state the definition and return to our question to give our proof.

Def. 1. Let $f\colon D \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be a function, the image of f, written $\mathrm{img}(f)$, is defined to be the subset of $\mathbb{R}^n$ given by

$$\mathrm{img}(f) :\equiv \{\vec{y} \in \mathbb{R}^n \mid \exists \vec{x} \in \mathbb{R}^m [f(\vec{x}) = \vec{y}]\} \quad .$$

We read this as "all those points $\vec{y}$ in $\mathbb{R}^n$ such that there exists a point $\vec{x}$ in $\mathbb{R}^m$ which is sent to $\vec{y}$ by f".

In our situation we have

$$\begin{aligned}\mathrm{img}(\mathbf{c}) &:\equiv \{(x,y,z) \in \mathbb{R}^3 \mid \exists t \in [a,b][\mathbf{c}(t) = (x,y,z)]\}\\ \mathrm{img}(\mathbf{d}) &:\equiv \{(x,y,z) \in \mathbb{R}^3 \mid \exists s \in [a_0,b_0][\mathbf{d}(s) = (x,y,z)]\}\end{aligned}$$

and so the question is asking us to show that these two sets are the same. To do that we'll have to prove both that $\mathrm{img}(\mathbf{c}) \subseteq \mathrm{img}(\mathbf{d})$ and that $\mathrm{img}(\mathbf{d}) \subseteq \mathrm{img}(\mathbf{c})$.

In the first case, given $(x,y,z) \in \mathrm{img}(\mathbf{c})$ we know that there is some $t \in [a,b]$ for which $\mathbf{c}(t) = (x,y,z)$. To show that $(x,y,z) \in \mathrm{img}(\mathbf{d})$ we must produce an $s \in [a_0,b_0]$ such that $\mathbf{d}(s) = (x,y,z)$. We can approach this either from the point of view of understanding, or from the point of view that there's only one we can do to turn t's into s's. Let's set $s :\equiv \alpha(t)$, and check what the value of $\mathbf{d}$ is at this s.

Recall that $\mathbf{d}(s)$ was defined to be $\mathbf{c}(t)$ a the *unique* value of t for which $\alpha(t) = s$. Well, we've defined s as $\alpha(t)$ so $\mathbf{d}(s)$ *is* $\mathbf{c}(t)$ by definition. That is to say, given $(x,y,z) \in \mathrm{img}(\mathbf{c})$ with $t \in [a,b]$ witnessing $\mathbf{c}(t) = (x,y,z)$ we see that $(x,y,z) \in \mathrm{img}(\mathbf{d})$ as $\mathbf{d}(\alpha(t)) = \mathbf{c}(t) = (x,y,z)$. Thus $\mathrm{img}(\mathbf{c}) \subseteq \mathrm{img}(\mathbf{d})$.

Now for the other inclusion, $\mathrm{img}(\mathbf{d}) \subseteq \mathrm{img}(\mathbf{c})$. Let us take $(x,y,z) \in \mathrm{img}(\mathbf{d})$, that is, $(x,y,z) \in \mathbb{R}^3$ is such that there is an $s \in [a_0,b_0]$ with $\mathbf{d}(s) = (x,y,z)$. In order to show that $(x,y,z) \in \mathrm{img}(\mathbf{c})$ we must produce a $t \in [a,b]$ satisfying $\mathbf{c}(t) = (x,y,z)$. How should we find this t? Recall that α is a bijection, and so for the $s \in [a_0,b_0]$ that we have, there exists a (unique) $t \in [a,b]$ with $s = \alpha(t)$. Let's take this t (which we may choose to call $\alpha^{-1}(s)$) and check the value that $\mathbf{c}$ takes on it: $\rfloor(t) = \mathbf{d}(\alpha(t)) = \mathbf{d}(s) = (x,y,z)$ where the first equality is the definition of $\mathbf{d}$, the second is by our definition of t in terms of s, and the third is by the assumption that $(x,y,z) \in \mathrm{img}(\mathbf{d})$. Thus we see that $(x,y,z) \in \mathrm{img}(\mathbf{c})$ and so $\mathrm{img}(\mathbf{c}) \subseteq \mathrm{img}(\mathbf{d})$ and overall $\mathrm{img}(\mathbf{c}) = \mathrm{img}(\mathbf{d})$. ∎